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LETTER TO THE EDITOR

Multiparameter R-matrices and their quantum groups

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Abstract. Multiparameter solutions of the Yang–Baxter equation associated with the groups of types A, B, C, D are given as generalizations of the well known one-parameter solutions. The R-matrix approach to quantum groups used here is related to the algebraic one of Rheshitikhin.

Much has been said about quantum groups based on the so-called standard solutions of the (quantum) Yang–Baxter equation (YBE) [1, 2] while on multiparameter quantum groups only a small number of publications is at hand [3–7]. In this letter it is shown that there exist natural multiparametric generalizations of the standard solutions.

A simple example [8] will be used to introduce notation and to distinguish different types of deformation parameters. The main part of this letter presents multiparametric R-matrix solutions of the YBE associated with the groups of types A, B, C, D, and shows how they are obtained by explicitly solving the YBE. These results are then related to the algebraic approach of Reshetikhin [5] by translating it to a 'Hopf algebra free' formulation of the R-matrix approach of Takhtajan and others [2]. We conclude with some remarks on the prospects of multiparameter quantum groups and its applications.

As in ordinary group theory there are three structures that are related by a quantum group: the group itself given as a matrix of non-commuting entries, a coordinate space transformed by the group action, and the Lie algebra associated to the group that is in general also deformed. The commutation relations can be represented most simply using the R-matrix. Let us consider the two-dimensional general linear group as an example ($R \in \mathbf{C}^{4\times4}$):

	notation	relations
quantum group	$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$	$R_{12}T_1T_2 = T_2T_1R_{12}$
quantum plane	$\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$	$(\hat{R}-1)(\boldsymbol{x}\otimes\boldsymbol{x}) = 0$ (1)
quantum Lie algebra	$T^{\pm}, H, K$	$R_{12}L_2^{\pm}L_1^{\pm} = L_1^{\pm}L_2^{\pm}R_{12}$ $R_{12}L_2^{+}L_1^{-} = L_1^{-}L_2^{+}R_{12}$

where $\hat{R} = PR$, P is the permutation matrix, the normalization s.t. $\hat{R}^{ii}_{ii} = 1$, and $L^{\pm}$ are matrix functionals of the generators $T^{\pm}, H, K$.

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It has been shown, that if one starts with choosing a q-plane

$$xy = q\, yx \tag{2}$$

and a deformation of the Lie algebra relations of gl(2) with parameter r

$$[T^+, T^-] = [H]_r \equiv \frac{r^H - r^{-H}}{r - r^{-1}} \qquad [T^\pm, H] = \pm T^\pm \qquad [K, ...] = 0 \tag{3}$$

this corresponds to the following R-matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/q & 0 & 0 \\ 0 & 1 - 1/r^2 & q/r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4}$$

The matrix functionals are in this case:

$$L^+ = \begin{pmatrix} p^{(H+K)/2} & 0 \\ (r - 1/r)\, \tilde{T}^+ & q^{(-H+K)/2} \end{pmatrix} \qquad L^- = \begin{pmatrix} q^{-(H+K)/2} & -(r - 1/r)\, \tilde{T}^- \\ 0 & p^{(H-K)/2} \end{pmatrix} \tag{5}$$

where $p \equiv r^2/q$, $\tilde{T}^+ \equiv (q/r)^{H/2}\, T^+$, $\tilde{T}^- \equiv T^-(q/r)^{H/2}$.

The group deformation then is to be:

$$\begin{aligned} ab &= \frac{r^2}{q}\, ba \qquad cd = \frac{r^2}{q}\, dc \qquad bc = \frac{q^2}{r^2}\, cb \\ ac &= q\, ca \qquad bd = q\, db \qquad ad - da = \frac{r^2 - 1}{q}\, bc \end{aligned} \tag{6}$$

which is the two-parameter deformation of GL(2). A second quantum plane of exterior variables ξ, η, that can be interpreted as differentials of x and y, is also determined:

$$\xi\eta = -\frac{q}{r^2}\, \eta\xi . \tag{7}$$

The determinant can be defined and becomes central for $q = r$ which allows us to define $\mathrm{SL}_q(2)$. (For details see [8], note the different notations for the second parameter r, $X = r^2$, or $p = r^2/q$.)

The underlying field for the coefficients has tacitly been assumed to be C and 'real' versions of the group can be obtained by giving an appropriate conjugation. There are, however, two ways to do this:

$$\begin{aligned} &(a) \qquad \overline{x} = y \qquad \overline{y} = x \\ &(b) \qquad \overline{x} = x \qquad \overline{y} = y . \end{aligned} \tag{8}$$

Since we require $\overline{xy} = \overline{y}\,\overline{x}$ the plane relations give:

$$xy = qxy \longrightarrow \quad \begin{aligned} &(a) \quad xy = \overline{q} yx \\ &(b) \quad xy = \tfrac{1}{\overline{q}} yx . \end{aligned} \tag{9}$$

For the group parameters conjugation has to be:

$$\begin{aligned} &(a) \qquad \overline{T} = STS = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &(b) \qquad \overline{T} = T \end{aligned} \tag{10}$$

and hence

$$\begin{aligned} ab = \frac{r^2}{q} ba \longrightarrow \quad &(a) \quad d = \frac{\overline{r^2}}{\overline{q}} dc \qquad \text{etc} \\ &(b) \quad ab = \frac{\overline{q}}{\overline{r^2}} ba \qquad \text{etc} \end{aligned} \tag{11}$$

yielding for the parameters

$$\begin{aligned} &(a) \qquad q, r \in \mathbf{R} \setminus \{0\} \\ &(b) \qquad q = \mathrm{e}^{\mathrm{i}\vartheta} \qquad r^2 = \mathrm{e}^{\mathrm{i}\Theta} \qquad \vartheta, \Theta \in \mathbf{R}. \end{aligned} \tag{12}$$

i.e. both deformation parameters are either real or pure phases. These groups may be denoted as $\mathrm{GL}_{r^2;q}(2,\mathbf{R})$ and $\mathrm{GL}_{\Theta;\vartheta}(2,\mathbf{R})$, respectively.

Unitary groups can be found by requiring

$$T^{+} = T^{-1} \qquad \text{i.e.} \qquad \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} = \begin{pmatrix} d & -pc \\ -b/p & a \end{pmatrix} \mathcal{D}^{-1} \tag{13}$$

giving the group $\mathrm{U}_{r^2;\vartheta}(2)$, where $q = r\,\mathrm{e}^{\mathrm{i}\vartheta}$, and $\mathrm{SU}_r(2)$, where $q = r \in \mathbf{R}$, with two and one real deformation parameters, respectively. ($\mathcal{D} = ad - q\,cb$ is the determinant [8].) This can be seen from, for example,

$$\begin{aligned} ac = q\,ca \longrightarrow b\mathcal{D}^{-1} d\mathcal{D}^{-1} = \overline{q} d\mathcal{D}^{-1} b\mathcal{D}^{-1} &\Rightarrow bd = \overline{q}\frac{q^2}{r^2}\,db \; (= q\,db!) \\ &\Rightarrow \overline{q}q = r^2 \,. \end{aligned} \tag{14}$$

Remark. In this paper we do not consider Hopf-$*$-algebra quantum groups of an extended algebra with barred and unbarred generators providing mixed commutation relations. For distinction we write $\mathrm{GL}(2)$ instead of $\mathrm{GL}(2,\mathbf{C})$. Such extensions are in general not difficult to find but not necessarily unique [9].

The procedure followed in the last section generalizes for $\mathrm{GL}(n)$ and with some restrictions also for $\mathrm{SL}(n)$, i.e. A_{n-1}. Similarly one can proceed for the classical series of semi-simple groups B_n, C_n, D_n, i.e. $\mathrm{SO}(2n+1)$, $\mathrm{Sp}(2n)$, $\mathrm{SO}(2n)$.

A_{n-1}—the groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$. As in the two-dimensional case the Lie algebra has still one deformation parameter; the quantum space relations, however, can be chosen to be deformed more generally:

$$x^i x^j = q_{ij}\, x^j x^i \qquad i < j \tag{15}$$

providing a total of $N_{\mathrm{GL}} = n(n-1)/2 + 1$ parameters (q_{ij} and r) for the corresponding R-matrix ($\mathbf{R} = R^{ij}_{kl}\, e_i{}^k \otimes e_j{}^l$):

$$R^{ij}_{kl} = \delta^i_k \delta^j_l \left(\delta^{ij} + \Theta^{ji} \frac{1}{q_{ij}} + \Theta^{ij} \frac{q_{ji}}{r^2} \right) + \delta^i_l \delta^j_k \, \Theta^{ij} \left(1 - \frac{1}{r^2} \right) \tag{16}$$

where Θ^{ij} equals 1 for $i > j$ and zero otherwise. Note, no summation over repeated indices is involved here. The group relations given by this R-matrix are not difficult to work out and can be found in [6].

To arrive at the quantized $\mathrm{SL}(n)$ from $\mathrm{GL}_{r^2;q_{ij}}(n)$ it is necessary to make the determinant $\mathcal{D}$ central before it can consistently be identified with unity. In the GL case the determinant has the following property [6]:

$$T^i_k \mathcal{D} = \frac{\prod_{\alpha=1}^{k-1} q_{\alpha k}}{\prod_{\beta=1}^{i-1} q_{\beta i}} \frac{\prod_{\gamma=k+1}^{n} (r^2/q_{k\gamma})}{\prod_{\delta=i+1}^{n} (r^2/q_{i\delta})} \mathcal{D}\, T^i_k \, . \tag{17}$$

In order to render the factor picked up to unity, one can simply choose all q's equal to r. This gives the well known one-parameter deformation $\mathrm{SL}_r(n)$. More generally the centrality of the determinant can be achieved by requiring

$$q_{1,i}\, q_{2,i} \cdots q_{i-1,i} \frac{r^2}{q_{i,i+1}} \cdots \frac{r^2}{q_{i,n}} = \text{constant} \tag{18}$$

for all $i = 1, \ldots, n$. This last requirement results in $n-1$ conditions among the q_{ij}'s in addition to determining constant $= r^{n-1}$. The number of deformation parameters has been reduced to $N_{\mathrm{SL}} = N_{\mathrm{GL}} - (n-1) = (n-1)(n-2)/2 + 1$.

To give an example consider $\mathrm{SL}_{r^2;q_{ij}}(4)$ with $q_{14} = r^3/q_{12}q_{13}$, $q_{24} = rq_{12}/q_{23}$ and $q_{34} = r/q_{13}q_{23}$.

There are again two ways to impose reality conditions on $\mathrm{GL}(n)$ ($i' \equiv n+1-i$):

$$\begin{array}{lllll} (a) & \overline{x}_i = x^i & \overline{T} = T & q_{ij} = \mathrm{e}^{\mathrm{i}\vartheta_{ij}} & r^2 = \mathrm{e}^{\mathrm{i}\Theta} \\ (b) & \overline{x}_i = x^{i'} & \overline{T^i_k} = T^{i'}_{k'} & \overline{q_{ij}} = q_{j'i'} & r^2 \in \mathbf{R}. \end{array} \tag{19}$$

giving $\mathrm{GL}_{\Theta;\vartheta_{ij}}(n,\mathbf{R})$ and $\mathrm{GL}_{r^2;\overline{q_{ij}}=q_{j'i'}}(n,\mathbf{R})$ and similarly for SL. Deriving the unitary versions is left to the reader.

B_n, C_n, D_n—orthogonal and symplectic groups. For the groups of types B, C, D an additional structure, the metric, has to be implemented in a consistent manner. The metric C defines a length $L = x^t C x$ and provides the orthogonality (or symplectic) relation

$$TC^{-1}T^tC = 1 = C^{-1}T^tCT \tag{20}$$

for quantum matrices T. Roughly speaking, it is the required centrality of the length forcing the metric to be anti-diagonal. (Clearly, it can be diagonalized in the classical limit):

$$C_{ij} = c_i\, \delta_{ij'} \qquad (j' \equiv n+1-j) \qquad \text{with } c_i\, c_{i'} = \pm 1. \tag{21}$$

(The last requirement is not necessarily so at this stage but has been made to facilitate the argument; it can be motivated, for example, by consistency of double conjugation, see below (38).) Hence also in the R-matrix a piece containing primed indices appears. We make the following ansatz (motivated by the 'standard' solutions):

$$R^{ij}_{kl} = r\left[\delta^i_k\delta^j_l\left(\frac{1}{q_{ij}}\right) + \left(1-\frac{1}{r^2}\right)\left(\delta^i_l\delta^j_k\,\Theta^{ij} + \delta^{ij'}\delta_{kl'}\,\Theta^i_k\,a^i_k\right)\right] \tag{22}$$

where Θ is as above and a^i_k are coefficients to be determined. Inserting the orthogonality condition, $T = C^{-1}(T^t)^{-1}CT$, into the group relations, $RT_1T_2 = T_2T_1R$, it follows that

$$R = C_1^{-1}R^{t_1}C_1 = C_2^{-1}(R^{-1})^{t_2}C_2 \tag{23}$$

implying for (22)

$$R^{ij}_{kl} = \frac{c_i\,c_j}{c_k\,c_l}R^{k'l'}_{i'j'}\ . \tag{24}$$

As in the GL case we have $q_{ii} = 1$ and $q_{ji} = r^2/q_{ij}$ and hence (24) yields

$$q_{ij} = \frac{r^2}{q_{ij'}} = \frac{r^2}{q_{i'j}} = q_{i'j'} \tag{25}$$

i.e. $q_{ij},\ i < j \leqslant n/2$, give all q's.

Having fixed the relations among the q's, one finds that the relations for the a^i_k's from the YBE decouple. From (24) we get

$$a^i_k = a^{k'}_{i'} \tag{26}$$

and the YBE provides

$$a^i_j a^j_k = -a^i_k \tag{27}$$

and

$$\begin{aligned} a^{n/2+1}_{n/2} &= -1 \qquad \text{or} \qquad +\frac{1}{r^2} && n \text{ even} \\ a^{(n+3)/2}_{(n-1)/2} &= -r && n \text{ odd.} \end{aligned} \tag{28}$$

In combination (26) and (27) are now a recursive set of relations that can be solved using the 'initial conditions' (28). Taking also into account that we have to meet the orthogonality relation (24), one finds $a^i_k = -\epsilon_i\epsilon_k r^{\hat{\imath}-\hat{k}}$ where $\hat{\imath}$ are integers determined by the above conditions and the ϵ's matter only in the symplectic case. Equation (28) yields:

$$(\hat{1}\ldots) = \begin{cases} (m-\frac{1}{2}, m-\frac{3}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, \frac{1}{2}-m) & \text{for } B_m \\ (\,m,\ m-1, \ldots, 1, -1, \ldots, 1-m,\ -m\,) & \text{for } C_m \\ (m-1, m-2, \ldots, 1, 0, 0, -1, \ldots, 1-m) & \text{for } D_m. \end{cases} \tag{29}$$

The c_i's of the metric have to be chosen as

$$C_{ij} = \epsilon_i \, r^{\hat{i}} \delta_{ij'} \qquad \text{with} \quad \epsilon_i = \begin{cases} +1 & \text{for } B_m \text{ and } D_m \\ +1 & \text{for } C_m \text{ and } i \leqslant m \\ -1 & \text{for } C_m \text{ and } i > m \end{cases} \tag{30}$$

thus $c_i \, c_{i'} = -1$ only for symplectic groups. In the notation of [2] we have:

$$\begin{aligned} \mathbf{R} = r \sum_{i \neq i'}^{n} e_{ii} \otimes e_{ii} &+ e_{(n+1)/2,(n+1)/2} \otimes e_{(n+1)/2,(n+1)/2} \\ &+ \sum_{\substack{i<j \\ i \neq j'}}^{n} \frac{r}{q_{ij}} e_{ii} \otimes e_{jj} + \sum_{\substack{i>j \\ i \neq j'}}^{n} \frac{q_{ji}}{r} e_{ii} \otimes e_{jj} + \frac{1}{r} \sum_{i \neq i'}^{n} e_{ii} \otimes e_{i'i'} \\ &+ \left(r - \frac{1}{r} \right) \left[\sum_{i>j}^{n} e_{ij} \otimes e_{ji} - sum_{i>j}^{n} r^{\hat{i}-\hat{j}} \, \epsilon_i \epsilon_j \, e_{ij} \otimes e_{i'j'} \right] \end{aligned} \tag{31}$$

with (29) and (30). The second term is only present for the B_m case.

Inspection of the classical limits shows the association with the indicated groups. It turns out that the non-diagonal parts of the multiparameter R-matrices coincide with the standard solutions (cf e.g. [2])—they coincide anyway for all $q_{ij} = r$—and also the characteristic equation depends only on the Lie algebra deformation parameter r and matches those for the standard solutions ($r = q$) if one normalizes $\hat{R}$ appropriately ($\hat{R}^{ii}_{ii} = 1$):

$$\begin{aligned} \left(\hat{R} - 1\right)\left(\hat{R} + \frac{1}{r^2}\right)\left(\hat{R} - \frac{1}{r^n}\right) &= 0 \qquad \text{for SO}(n) \\ \left(\hat{R} - 1\right)\left(\hat{R} + \frac{1}{r^2}\right)\left(\hat{R} + \frac{1}{r^{n+2}}\right) &= 0 \qquad \text{for Sp}(n)\ . \end{aligned} \tag{32}$$

Remark: Strictly speaking, the orthogonal groups are just $\mathrm{O}(n)$ rather than $\mathrm{SO}(n)$, i.e. the determinant is ± 1. It is, however, possible to use the isomorphy $\mathrm{SO}(n) \simeq \mathrm{O}(n)/Z_2$ by identification of pairs of quantum matrices having different sign of the determinants, e.g. $T \sim \mathrm{diag}(-1, 1, 1, \ldots)\, T$. Alternatively one could require the determinant condition explicitly.

Example: $\mathrm{SO}_{r^2;q}(4)$. The two-parameter orthogonal group in four dimensions illustrates the greater generality of multiparameter quantum groups. Since the characteristic equation coincides with that of the standard solutions, so does the projector decomposition of $\hat{R}$ [2]. The relations for the quantum vector space are:

$$\begin{aligned} &x_1 x_2 = q\, x_2 x_1 \qquad && x_1 x_3 = \frac{r^2}{q}\, x_3 x_1 \qquad && x_2 x_3 = x_3 x_2 \\ &x_2 x_4 = q\, x_4 x_2 \qquad && x_3 x_4 = \frac{r^2}{q}\, x_4 x_3 \qquad && x_1 x_4 = x_4 x_1 - \left(r - \frac{1}{r} \right) x_2 x_3 \end{aligned} \tag{33}$$

and for the exterior coordinates, $(\xi^i)^2 = 0$, we find:

$$\begin{aligned} &\xi_1\xi_2 + \frac{q}{r^2}\,\xi_2\xi_1 = 0 \qquad \xi_1\xi_3 + \frac{1}{q}\,\xi_3\xi_1 = 0 \qquad \xi_2\xi_3 + \xi_3\xi_2 = \left(r - \frac{1}{r}\right)\xi_1\xi_4 \\ &\xi_2\xi_4 + \frac{q}{r^2}\,\xi_4\xi_2 = 0 \qquad \xi_3\xi_4 + \frac{1}{q}\,\xi_4\xi_3 = 0 \qquad \xi_1\xi_4 + \xi_4\xi_1 = 0. \end{aligned} \tag{34}$$

The metric (21)

$$C = \begin{pmatrix} & & & r^{-1} \\ & & 1 & \\ & 1 & & \\ r & & & \end{pmatrix} \tag{35}$$

gives the central length

$$L = r^{-1}x_1x_4 + x_2x_3 + x_3x_2 + r x_4x_1 = (r + r^{-1})(x_1x_4 + r\,x_2x_3)\ . \tag{36}$$

Interestingly enough, the two-parameter R-matrix for SO(4) cannot be constructed as the tensor product of two SL(2) (unless $q = r$) [10]:

$$R(\mathrm{SO}_{r^2;q}(4)) \neq R(\mathrm{SL}_p(2)) \otimes R(\mathrm{SL}_{p'}(2)). \tag{37}$$

Real versions. Conjugation can be defined trivially $\overline{x} = x$, $\overline{T} = T$, or with help of the metric

$$\overline{x^i} = \overline{x}_i = C^t_{ik}\,x^k \qquad \text{i.e.} \qquad \overline{x}_i = c_{i'}\,x^{i'} \tag{38}$$

$$T^+ = T^{-1} \qquad \text{i.e.} \qquad \overline{T} = (T^{-1})^t\ . \tag{39}$$

Compatibility with the quantum group relations can be checked by computing the adjoint of

$$\hat{R}\cdot(T\otimes T) = (T\otimes T)\cdot\hat{R}\ . \tag{40}$$

One finds

$$\check{R}^+\cdot(T\otimes T) = (T\otimes T)\cdot\check{R}^+ \tag{41}$$

where $\check{R} = RP$. This holds if the R-matrix obeys

$$\overline{(R^{lk}_{ij})} = R^{ji}_{kl} \qquad \text{i.e.} \qquad \overline{q_{ij}}\,q_{ij} = r^2 \in \mathbf{R} \qquad \text{or} \qquad q_{ij} = r\,\mathrm{e}^{\mathrm{i}\vartheta_{ij}}. \tag{42}$$

The plane deformation parameters q_{ij} exhibit a deformation structure which one could call 'radial quantization', where the parameter r provides the modulus for all q's combined with an independent phase. Hence, taking real deformation parameters its number stays the same as without reality condition.

In order to investigate the signature of the metric one can define 'real' coordinates:

$$y^k = \begin{cases} x^k + c_{k'}x^{k'} & k \leqslant \frac{n+1}{2} \\ i(c_{k'}x^{k'} - x^k) & k > \frac{n+1}{2} \end{cases} \qquad \Longrightarrow \qquad \overline{y^k} = y^k\ . \tag{43}$$

Under this transformation the metric becomes

$$C \longrightarrow \tilde{C} = \begin{pmatrix} 1+c_1^2 & & i\left(c_1 - \frac{1}{c_1}\right) \\ & \cdot \cdot \cdot & \\ i\left(c_n - \frac{1}{c_n}\right) & & 1+c_n^2 \end{pmatrix}. \tag{44}$$

Since for $r \rightarrow 1$ all c_i become unity, this reality condition gives the quantum groups $SO_{r^2;\vartheta_{ij}}(n, \mathbf{R})$.

In the other case $\overline{x} = x$ (thus the coordinates are already 'real') diagonalization of the antidiagonal metric makes us arrive at $SO_{\Theta;\vartheta_{ij}}(n/2, n/2)$ and $SO_{\Theta;\vartheta_{ij}}((n+1)/2, (n-1)/2)$ for n even and odd, repectively. Constructing deformed groups of other signature is a more involved topic that needs more complicated R-matrices. For the Lorenz group it is not yet clear whether one or two independent parameters can be found.

Reshetikhin has developed a derivation of multiparameter R-matrices from the standard one-parameter versions employing the universal R-matrix living in the tensor product of an algebra A with itself and having Hopf algebra properties [5]. It can be shown that a second element of the tensor algebra, given in terms of the generators of the Cartan subalgebra of the associated deformed Lie algebra, provides a transformation yielding a new universal R-matrix. By representation the numerical multiparameter R-matrix is constructed.

It is easy, however, to translate this algebraist approach to a simple matrix formalism on the lines of Takthajan [2] and others omitting the detour to the universal R-matrices [13]: The definition of a Hopf algebra with comultiplication Δ and an universal $R \in A \otimes A$ s.t.

$$(\Delta \otimes \mathrm{id})R = R_{13}R_{23} \qquad (\mathrm{id} \otimes \Delta)R = R_{13}R_{12} \qquad \sigma \circ \Delta(a) = R\Delta(a)R^{-1} \tag{45}$$

implies the Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{46}$$

which contains all information we need in the R-matrix approach to quantum groups.

Reshetikhin requires for an additional element $F \in A \otimes A$

$$(\Delta \otimes \mathrm{id})F = F_{13}F_{23} \qquad (\mathrm{id} \otimes \Delta)F = F_{13}F_{12} \qquad F_{12}F_{21} = \mathbf{1} \tag{47}$$

and also the YBE

$$F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12}. \tag{48}$$

Then $R^F \equiv F^{-1}RF^{-1}$ is also a solution of the YBE.

Applying $(\sigma \circ \Delta \otimes \mathrm{id})$ on F in two different ways, firstly letting σ simply interchange the embeddings and secondly using the last equality of (45), one finds:

$$\begin{aligned}(\sigma \circ \Delta \otimes \mathrm{id})(F) &= \sigma_{12}(F_{13}F_{23}) = F_{23}F_{13} \\ &= R_{12}F_{13}F_{23}R_{12}^{-1}\end{aligned} \tag{49}$$

i.e.

$$R_{12}F_{13}F_{23} = F_{23}F_{13}R_{12} \tag{50}$$

and similarly using $(\mathrm{id} \otimes \sigma \circ \Delta)(F)$

$$F_{12}F_{13}R_{23} = R_{23}F_{13}F_{12}\ . \tag{51}$$

Like the YBE for R, also those for F and for mixed R and F can now simply be considered as matrix equations. With this 'translation' it needs just a triangular $(F_{12}F_{21} = 1)$ solution of the YBE that obeys also (50) and (51) in order to associate a new R-matrix to a given one. (At this stage the construction obviously becomes circular since we need a solution of the YBE in order to find a new one; it turns out, however, that rather trivial solutions help.)

If we choose $F = \mathrm{diag}(f_{11}, f_{12}, \ldots, f_{nn})$ with $f_{ij}f_{ji} = 1$, it is easy to show that

$$F^{-1}\ R(GL_r)\ F^{-1} = R(GL_{r^2;q_{ij}}) \qquad q_{ij} \equiv r f_{ij}^2 \tag{52}$$

i.e. the standard one-parameter deformation is transformed to the multiparameter deformation (16). (In terms of $\hat{R}$ this is a similarity transformation that, however, does not decompose into a simple tensor product $F \neq f \otimes g$.)

By the same means the multiparameter versions of the R-matrices of types B, C, D are related to the standard solutions since, having an additional term of type $\delta^{ij'}\delta_{kl'}$ equations (50) and (51) require in addition

$$f_{ij} = \frac{1}{f_{ij'}} = \frac{1}{f_{i'j}} = f_{i'j'} \tag{53}$$

which turns out to be equivalent to (25).

This R-matrix formulation also provides a nice way to verify the results of the last section. Using (49), (50) and (51) one can reduce the YBE for R^F to those for R and F alone.

Remark. There are nonetheless additional one-parameter solutions of the YBE that are related to the 'standard' one by a non-diagonal matrix [12,13]. Since the entry structure of these R-matrices is more complicated, more restricting relations of type (53) arise making multiparameter versions almost always impossible.

In accordance with the abstract treatment of Reshetikhin we have explicitly shown, that for groups of type A, B, C, D that $r(r-1)/2+1$ quantization parameters emerge from solving the YBE if r is the rank of the group. It is the maximal number of deformation parameters for quantum groups with the same structure of commutation relation as the standard solutions.

To give an idea of the coherence of multiparameter quantum groups, we note that the fact that GL(n) and Sp($2n$) are of the same rank corresponds to the fact that a $2n$-dimensional phase space built from a n-dimensional linear space (cf Zumino's construction [14]) allow for symplectic transformations also in the deformed case not restricting the possible multiparametric deformation structure.

A many-parameter deformation of a group provides many one-parameter deformations. Thus this letter does not intend to persuade the reader to use always the most general deformation structure possible but rather points out that different parameter fixings lead to different structures with few or only one parameter having different properties. See, for example, [15] exhibiting that it is the deformation $GL_{1;q}$ rather than the 'standard' deformation $GL_{q^2;q}$ that seems to be suitable to define a path integral.

In general, it will depend on the physical application which deformation structure is appropriate. Hoping that the 'standard' one will do is, however, not justified.

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